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THE ISOPERIMETRIC PROBLEM OF PROFILING OF THE OPTIMUM CLEARANCE OF AN INFINITE PLANE SLIDER BEARING†

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The isoperimetric problem (IP) of profiling the optimum clearance between a plane support surface and an infinite cylindrical (plane) slide is formulated and solved in the incompressible fluid approximation. If the maximum of the carrying capacity coefficient C_N is realized in the well-known Rayleigh problem (RP), where L in the IP the minimum friction is ensured for the given value of C_N. The structure of the optimum solution is explained and it is established that if C_N is less than the coefficient C_{NR} corresponding to the RP, then the clearance height h is a continuous function of the x coordinate measured along the support surface. In the general case the optimum function $h = h(x)$ may contain segments of four kinds. Two of them, $h = 1$ and $h = H > 1$, are the boundary extremum segments (BES1 and BESH), which appear due to the fact that h has upper and lower bounds. The other two segments are bilateral extremum segments. TES1 is similar to the TES in Rayleigh's problem, in which $h = h_1$, where $1 < h_1 < H$. TES2 appears only in the IP. It has a negative slope and connects BES1 with BESH or TES1. As $C_N \to C_{NR}$ the slope of TES2 approaches minus infinity, and the segment itself turns into a step, i.e. into the well-known discontinuity of h in the RP. © 1998 Elsevier Science Ltd. All rights reserved.

The problem of profiling the optimum clearance in the approximation of lubrication theory was considered by Rayleigh [1]. In the problem of determining the clearance of an infinite cylindrical slider bearing that gives the maximum of C_N, solved by Rayleigh within the framework of an incompressible viscous fluid, the optimum clearance is piecewise constant with one step. Over the initial segment TES1 the clearance height $h \equiv h_1 > 1$ satisfies Euler's equation. The terminal segment $h \equiv 1$, where h is measured relative to the maximum admissible height h_m in the formulation of the problem, is a boundary extremum segment (BES1). The stepwise solution of the RP also applies to a polytropic gas for a variable clearance height over TES1 [2, 3]. In recent years the RP has also been solved in the three-dimensional formulation [4–9]. The variational problems solved in the papers mentioned above are generalizations of the RP on the maximum of C_N and its characteristic singularity, namely, the stepwise distribution of h.

Along with the RP, it makes sense to consider other variational problems, in particular, the problem of minimizing the drag C_D with C_N fixed. This problem is formulated and solved below.

For a plane slide the IP was considered in [10], where the carrying capacity N was fixed instead of C_N when there were no constraints on h, and the role of h_m was played by h_N, chosen to ensure the given value of N. In this formulation the IP was reduced to the minimization of $R^0 = C_D/\sqrt{C_N}$ for $h(x) \geqslant 0$. The problem of optimizing the clearance to obtain the minimum of $R = C_D/C_N$ was considered in the same paper. The relation between the results obtained in [10] and the present research is discussed at the end of the paper.

1. Let xyz be Cartesian coordinates connected with a cylindrical slide that is infinite along the z axis and moves over the plane $y = 0$ in the negative direction of the x axis at constant velocity $-U$. In this system of coordinates the slide is at rest, while the plane $y = 0$ moves with velocity $U > 0$, as shown in Fig. 1(a). The height $y = h(x)$ of the clearance, which constitutes the support of the slide if, may in general have a step at $x = x_d$. We shall denote the values of the variables at the points $i, f, d, \ldots$ by the appropriate subscripts. If the variables have a discontinuity at d, then we shall use an additional subscript minus (plus) four values to the left (right) of d. As the scale of x and h we take the width L of the slide and its minimum attainable height h_m. The pressure is fixed at $x = 0$ and $x = 1$. As the scale of pressure p and the x-component of the velocity vector u we take pU^2 and U, where ρ in the fluid density. Then (p_0 is a known constant)

$$u(x,0) = 1, \quad u(x,h) = 0, \quad p(0,y) = p(1,y) = p_0 \tag{1.1}$$

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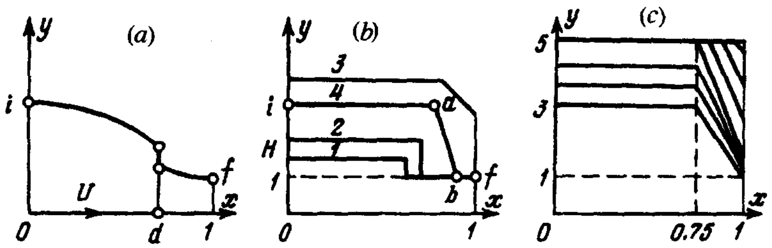

Fig. 1.

By the estimates of lubrication theory [11] the pressure p depends on y and the inertial terms in the projection of the equation of motion onto the x axis are small compared with the viscous ones. Integrating it twice with respect to y, taking the first two conditions in (1.1) into account, we find

$$\gamma u(x,y) = \gamma - \frac{\gamma + 3p'(x)h^2(x)}{h(x)} y + 3p'(x)y^2, \quad \gamma = \frac{6L\mu}{h_m^2 \rho U} \tag{1.2}$$

Here and henceforth differentiation with respect to x is denoted by a prime. The variables on the right-hand side of the dimensionless complex γ are dimensional ones and μ is the viscosity, taken to be constant.

We integrate (1.2) with respect to y from $y = 0$ to $y = h$, using the conditions for u from (1.1) and the fact that the integral on the left-hand side in (1.2), which is proportional to the flow rate $q/2$ through the clearance, is constant. As a result, we obtain

$$\pi' = (h-q)/h^3 \quad (\pi = p/\gamma) \tag{1.3}$$

Let N be the carrying capacity of the slide and let D be the drag force caused by it, i.e. the sum of the friction force and the integral of the pressure forces (the difference $p - p_0$) over y along the support, including a possible jump at $p = p_d$. We have

$$C_N = \frac{N}{\gamma L \rho U^2} = \int_0^1 (\pi - \pi_0)dx, \quad C_D = \frac{D}{\gamma h_m \rho U^2} = \frac{1}{2}\int_0^1 \left(\frac{1}{3h} + \pi' h\right)dx \tag{1.4}$$

In the RP we shall seek a height $h = h(x)$ that realizes the maximum of C_N for the function π defined by (1.3) with boundary conditions $\pi(0) = \pi(1) = \pi_0 = p_0/\gamma$. In the IP $C_N \leqslant C_{NR}$ is fixed and C_D is minimized. In both problems the height h has an upper and lower bound, i.e.

$$1 \leqslant h(x) \leqslant \mathrm{H} \tag{1.5}$$

according to the choice of the scale of h for a given constant $H \geqslant 1$.

2. To derive the optimality conditions we write down the Lagrange functional

$$J = \alpha C_D + \beta C_N + \Lambda, \quad \Lambda = \int_0^1 \lambda(x)(h - \pi' h^3 - q)dx$$

where λ is a variable Lagrange multiplier, $\alpha = 0$ in the RP, $\alpha = 1$ in the IP, β is a constant Lagrange multiplier and the expression in brackets in the integrand is equal to zero by (1.3). For an admissible variation, the variations of J and the optimizing functional in (1.4) are identical for any bounded Lagrange multipliers. Therefore, for the optimum clearance

$$\delta J = \delta C_N \leqslant 0 \text{ in the RP}, \quad \delta J = \delta C_D \geqslant 0 \text{ in the IP} \tag{2.1}$$

for any δh satisfying (1.5). Varying J, we take into account that π is fixed at the entry and at the exit of the clearance and h is continuous over the sections of possible height jumps. As a result, for any (not necessarily optimum) clearance height $h(x)$ and, so far, arbitrary constraints $\lambda(x)$, and in the IP also μ, we have

$$\delta J = \{\tfrac{1}{2}\alpha(h_- - h_+) + [(\lambda h^3)_+ - (\lambda h^3)_-]\}_d \Delta\pi_d + X_d \Delta x_d -$$

$$-\Delta q \int_0^1 \lambda dx + \int_0^1 (A^h \delta h + A^\pi \delta\pi) dx$$

$$X = \frac{\alpha}{6}\left(\frac{1}{h_-} - \frac{1}{h_+}\right) + \lambda_-(h-q)_- + \lambda_+(h-q)_+$$

$$3h^3 A^h = AB, \quad A = h - 3q/2, \quad B = \alpha - 6\lambda h^2, \quad 2A^\pi = 2\beta - (\alpha h - 2\lambda h^3)' \tag{2.2}$$

Here $\Delta\pi_d$ and Δx_d are the differences between the values of π and x over the sections corresponding to the jump of h for the varied and non-varied clearances, while $\delta\pi$ and δh are the differences between the values of π and h over these sections for fixed x. The coefficient X is transformed using (1.3).

Using the arbitrariness in the choice of λ, we make A^π equal to zero. This leads to the equation

$$(\alpha h - 2\lambda h^3)' = 2\beta \tag{2.3}$$

which holds for any clearance over the continuity segments of h. At the possible jump sections of h we can obtain conditions connecting λ_{d-} with λ_{d+} by equating the coefficient of $\Delta\pi_d$ to zero. This gives

$$\alpha(h_- - h_+)_d + 2[(\lambda h^3)_+ - (\lambda h^3)_-]_d = 0 \tag{2.4}$$

By (2.4) the expression in brackets in (2.3) is continuous. Equations (2.3) and condition (2.4) at the points d are insufficient for determining λ. The missing condition can be obtained by equating the coefficient of Δq to zero

$$\int_0^1 \lambda dx = 0 \tag{2.5}$$

Suppose that $h(x)$ is a given continuous or discontinuous function. For this function we can integrate (1.3) with given $\pi(0) = \pi_0$ and choose q to satisfy the condition $\pi(1) = \pi_0$, solving the direct problem of lubrication theory. Next, the constant λ_0 in the solution of (2.3), which depends linearly on $\lambda_0 = \lambda(0)$, is found from (2.5) by solving the adjoint problem for λ. Then we obtain

$$\delta J = X_d \Delta x_d + \int_0^1 A^h \delta h dx \tag{2.6}$$

$$X = \frac{\alpha(h_+ - h_-)}{6h_+h_-^3}[3qh_+ - h_-(3h_+ - h_-)] + \frac{\lambda_+}{h_-^3}(h_+ - h_-)f =$$

$$= \frac{\alpha(h_+ - h_-)}{6h_-h_+^3}[3qh_- - h_+(3h_- - h_+)] + \frac{\lambda_-}{h_+^3}(h_+ - h_-)f$$

$$f = h_+h_-(h_+ + h_-) - (h_+^2 + h_+h_- + h_-^2)q$$

$$3h^3 A^h = AB, \quad A = h - 3q/2, \quad B = \alpha - 6\lambda h^2$$

Both equivalent representations of X are obtained from (2.2) after eliminating λ_- or λ_+ using (2.4).

In the Rayleigh problem with $\alpha = 0$ and $\beta = 1$, analysis of (2.6) indicates that the optimum clearance may consist of the boundary extremum segments BES1 and BESH, where $h = 1$ and $h = H$, and the bilateral extremum segment TES1. On TES1 the height h can be determined by equating A^h or A to zero, i.e. by the equality

$$2h - 3q = 0 \tag{2.7}$$

On BES1 the admissible $\delta h \geqslant 0$ and on BESH $\delta h \leqslant 0$. In the RP $\delta J = \delta C_N$ and the admissible variation of the optimum clearance can only lead to a reduction of δC_N. Therefore the optimality conditions on these segments take the form

$$\lambda(2 - 3q) \geqslant 0, \quad h = 1; \quad \lambda(2H - 3q) \leqslant 0, \quad h = H \tag{2.8}$$

Different segments may be joined with or without a discontinuity of h. For the optimum "discontinuous" joints both in RP and the IP it is necessary for the coefficient of Δx_d in (2.6) to be zero over the section where h has a jump. In the RP with $\alpha = 0$ this condition shows that a "discontinuous" joint between TES1 and any boundary extremum segment is possible if

$$\lambda_{d-} = \lambda_{d+} = 0 \tag{2.9}$$

over the section where h is discontinuous. By (2.7) and (1.3)

$$h = 3q/2, \quad \pi' = 4/(27q^2) \tag{2.10}$$

on TES1. It follows that $h \equiv$ const on TES1 and π increases linearly with x. Therefore, by (2.8)–(2.10) in the RP the optimum height of the clearance $h(x)$ has one discontinuity which is located either between BESH and BES1 when $\mathrm{H} \approx 1$ and TES1 is simply not present, or between TES1 and BES1. Such clearances are indicated by the numbers 1 and 2 in Fig. 1(b).

In the IP $\alpha = 1$, $a^h = 0$ not only for $A = 0$, but also for $B = 0$, which leads, as before, to (2.7) and to TES1, but also due to the vanishing of B. In the second case

$$6\lambda h^2 = 1 \tag{2.11}$$

Therefore TES2 is possible in the IP. By (2.3) and (2.11) it is defined by the equation

$$h' = 3\beta \tag{2.12}$$

By analogy with (2.8) the inequalities

$$(1 - 6\lambda)(2 - 3q) \geqslant 0, \ h = 1; \quad (1 - 6\lambda \mathrm{H}^2)(2\mathrm{H} - 3q) \leqslant 0, \ h = \mathrm{H} \tag{2.13}$$

should be satisfied in the IP. If λ_+ or λ_- from (2.11) are substituted into X_d with $\alpha = 1$ from (2.6), then it can be shown that $X_d = 0$ only if $h_{d+} = h_{d-}$, i.e. TES2 joins other segments in a continuous manner.

3. We begin the task of solving the IP with the case when $C_N = 0$. The latter equality holds for any constant clearance ($h =$ const), while C_D decreases as the height of the clearance increases. Therefore, by the upper bound of h, for $C_N = 0$ the solution of the IP is given by

$$h(x) = \mathrm{H} \tag{3.1}$$

Under the boundary conditions $\pi(0) = \pi(1) = \pi_0$, from (1.3) and (3.1) it follows that

$$\pi(x) \equiv \pi_0 \tag{3.2}$$

For the solution of (3.1) to yield the minimum of C_D the second condition in (2.13) should be satisfied. Since $h \equiv \mathrm{H} = q$ and $2\mathrm{H} - 3q = -q < 0$ by (1.3), (3.1) and (3.2), the condition can be reduced to

$$1 - 6\lambda \mathrm{H}^2 \geqslant 0 \ \text{for} \ 0 \leqslant x \leqslant 1 \tag{3.3}$$

Determining $\lambda(x)$ from (2.3) with h from (3.1) and from condition (2.5), we obtain

$$\lambda(x) = \beta(1 - 2x)/(2\mathrm{H}^3) \tag{3.4}$$

By (3.4) λ is a linear function of x. Since (3.3) involves a linear expression in λ, it is satisfied if the left-hand side in (3.3) is non-negative for $x = 0$ and 1. Therefore (3.3) holds for $-H/3 \leqslant \beta \leqslant \mathrm{H}/3$. The exact value of β determines a continuous transition from $C_N = 0$ to $C_N > 0$. For this to be possible the arbitrarily small positive coefficient C_N can be realized by introducing a small TES2 in the neighbourhood of $x = 1$. It follows that for $C_N = 0$ inequality (3.3) should become an equality when $x = 1$. Because of this

$$\beta = -\mathrm{H}/3 < 0 \tag{3.5}$$

corresponds to the case $C_N = 0$ and a positive value of C_N in the neighbourhood of $C_N = 0$ gives a clearance consisting of a BESH at the entrance (for $0 \leqslant x \leqslant x_a < 1$) and a narrowing TES2 at the exit (profile 3 in Fig. 1b). Now, by (1.3) the pressure over the BESH increases along with x, since q is smaller than for $h \equiv H$ because of the narrowing end segment TES2. On BESH π and, by (2.3), also λ are linear functions of x. At a, where BESH and TES2 join one another continuously, $h_a = H, B_a = 0$ with B from (2.6) and $\lambda_a = 1/(6H^2)$. To the right of this point $\lambda = 1/(6h^2)$.

In accordance with (2.12) and (3.5), at the time when TES2 appears, it is, as it should be, a narrowing segment. Finding h from (3.5) as a function of x, x_a, β and H, we can substitute it into (1.3) and, using the pressure π_α already found as a function of q and x_a, we can find π everywhere for $x_a \leqslant x \leqslant 1$. The condition $\pi(1) = \pi_0$ together with the integral equality (2.6), on substituting into the latter $\lambda(x)$, determined as described above, gives two relations between q, x_a, β and H. If we introduce $\beta° = 3\beta/\mathrm{H}$ and $q° = q/\mathrm{H}$, these relations take the form

$$2(q°-1)x_a = \frac{1-x_a}{[1+\beta°(1-x_a)]^2}[2(1-q°)+\beta°(2-q°)(1-x_a)]$$

$$x_a+\beta° x_a^2+\frac{1-x_a}{1+\beta°(1-x_a)}=0, \quad \beta°=\frac{3\beta}{\mathrm{H}}, \quad q°=\frac{q}{\mathrm{H}} \tag{3.6}$$

The first of these is a "condition for the pressure", i.e. a condition stipulating that there is no pressure drop in the clearance. In what follows the second condition, which is a consequence of condition (2.5) for λ, will be called the "condition for λ". On changing from β to q to $\beta°$ and $q°$ in these conditions, H disappears from (3.6). This was to be expected, because up to now, while there is no TES1 in the solution, it has been natural to take H rather than h_m as the characteristic clearance height. After that, H manifests itself only through the constant γ with h_m replaced by H.

By (3.6) $\beta°$ and $q°$ are functions of x_a alone. By (3.5)

$$\beta°(1)=-1, \quad q°(1)=1 \tag{3.7}$$

Solving the second equation in (3.6) with respect to $\beta°$ and selecting from the two roots the one that gives the value from (3.7) as $x_a \to 1$, we obtain $\beta°$ and, substituting the resulting expression into the first equation in (3.6), also $q°$. We have

$$\beta°=\frac{\sqrt{4x_a-3}-1}{2x_a(1-x_a)}, \quad q°=2\frac{3x_a-2+x_a\sqrt{4x_a-3}}{5x_a-3+(3x_a-1)\sqrt{4x_a-3}} \tag{3.8}$$

By (3.8) $\beta°$ varies from -1 to $-8/3$ as x_a varies from 1 to 3/4. When $x_a < 3/4$ the radicand in (3.8) is negative. However, since $q° = 2/3$ for $x_a < 3/4$ and $\beta° = -8/3$, at the time when x_a reaches the value 3/4 the factor $A = 2\mathrm{H} - 3q$ in A^h from (2.6) becomes equal to zero and the first inequality in (2.13) becomes an equality over the whole BESH. Then the optimum clearance "freezes" in the sense that the optimum $h(x)$ corresponding to larger values of C_N is obtained for fixed $x_a < 3/4$, $\beta° = -8/3$ and $q° = 2/3$. Now, however, in (3.6) β and q correspond not to the given value of H, but to a clearance height h_1 smaller than H over the horizontal entry segment TES1. As h_1 decreases, so do flow rate $q = 2h_1/3$ and the modulus $3\beta = -8h_1/3$ of the slope of TES2, while the pressure in the clearance increases. Then $h°(x) = h/h_1$ and the coefficients $C_{N1} = N/(\gamma_1 L\rho U^2)$ and $C_{D1} = D/(\gamma_1 h_1 \rho U^2)$ with $\gamma_1 = 6L\mu/(h^2{}_1\rho U)$ do not change. In particular, $h°_f = 1/3$. Recalling definitions (1.4), we find

$$C_N = h_m^{\circ 2} C_{N1}, \quad C_D = h_m^\circ C_{D1}, \quad h_m^\circ = h_m/h_1 \tag{3.9}$$

For $h_1 \leqslant \mathrm{H}$ these equalities hold until the value of C_N for which $h_f = h_f^\circ h_1 = h_1/3 = 1$ is reached, i.e. the clearance attains the minimum admissible value at the exit of the narrowing TES2. But if we move in the direction of decreasing C_N, then for $\mathrm{H} \to \infty$ the range of validity of (3.9) and the "self-modelling" (in the aforesaid sense) optimum clearance on the plane of the coefficients C_N and C_D reaches the origin of the system of coordinates. In a finite neighbourhood of the origin, up to the values $C_N = C_{N3}$ and $C_D = C_{D3}$ corresponding to $H = 3$, the optimum value of C_D and h_1 are, by (3.9), related to C_N by the following formulae (the values of C_{N3} and C_{D3} are given in the next section)

$$C_D = k_D\sqrt{C_N}, \quad h_1 = k_h/\sqrt{C_N}; \quad k_D = C_{D1}/\sqrt{C_{N1}} = C_{D3}/\sqrt{C_{N3}}, \quad k_h = 3\sqrt{C_{N3}} \tag{3.10}$$

When there is no upper bound on h ($\mathrm{H} = \infty$) the optimum clearance consisting of two two-sided extremum segments, namely, a horizontal TES1 for $0 \leqslant x \leqslant 3/4$ and a rectilinear narrowing TES2 for $3/4 \leqslant x \leqslant 1$, is realized up to the value $h_1 = 3$. Formulae (3.10) hold up to the same value. Figure 1c shows the evolution of the optimum clearances for $\mathrm{H} = 5$ and $3 \leqslant h_1 \leqslant 5$.

Increasing C_N further involves clearances with three segments (profile 4 in Fig. 1b): a horizontal TES1 ia with $h = h_1 < 3$, an inclined TES2 ab, and BES1 with $h = 1$. The segment ab connects points with coordinates $x = x_a$, $h = h_a = h_1$, $x = x_b \geqslant x_a$ and $h_b = 1$. The last condition together with the equation of an inclined segment enables us to express x_b in terms of h_1, β and x_a

$$x_b = x_a + (1 - h_1)/(3\beta) \tag{3.11}$$

Equality (3.11) holds not only for the horizontal segment TES1, where $h_1 = 3q/2$, but also for BESH. The latter may occur for $H < 3$.

The condition for the pressure and the condition for λ for a clearance consisting of two horizontal BES's and an inclined TES2 connecting them takes the form

$$1 - h_1\omega^2 + \varepsilon[1 + (3 - 2h_1)h_1\omega] + \varepsilon^2(1 - h_1)^3 = 0$$

$$2(h_1 - q) + 2(1 - q)h_1^2\omega + \varepsilon(h_1 - 1)^2[2h_1(1 - q) - q] = 0 \tag{3.12}$$

$$\omega = h_1(1 - x_a)/x_a, \quad \varepsilon = h_1/(3\beta x_a)$$

on eliminating x_b using (3.11). Eliminating ε from (3.12) we obtain the quadratic equation

$$a\omega^2 - 2b\omega + c = 0 \tag{3.13}$$

$$a = -h_1[h_1^2(3q^2 - 6q + 4) + 2h_1q(q - 2) + q^2] = -4h_1^3(h_1 - 1)(4 - 3h_1)/9$$

$$b = h_1[2h_1^2(q^2 - 2q + 2) + h_1q(q - 6) + 3q^2] = 4h_1^3(h_1 - 1)(2h_1 - 3)/9$$

$$c = 4h_1^4(q - 1)^2 - 4h_1^3(q^2 - 3q + 3) - h_1^2(3q^2 - 12q - 4) - 6h_1q(q + 1) + 3q^2 =$$

$$= 4h_1^2(h_1 - 1)[4(h_1 - 1)^3 + 1]/9$$

where the second expressions for a, b and c are obtained from the previous ones by the substitution $q = 2h_1/3$, which corresponds to a clearance with TES1.

In the given solution $h_1 = 3$ and $q = 2h_1/3 = 2$ correspond to $x_a = 3/4$ and $\omega = 1$. This determines the choice of the sign in the solution of the quadratic equation (3.13). Using this choice for $q = 2h_1/3$, we have

$$\omega = \frac{h_1(2h_1 - 3) - 2\sqrt{3h_1}(h_1 - 1)^2}{h_1(4 - 3h_1)} \tag{3.14}$$

For a sufficiently stringent restriction on h when $h_1 = \mathrm{H} < 3q/2$, which occurs for $H < 3$, (3.14) must be replaced by

$$\omega = (b - d)/a \tag{3.15}$$

$$d = (2h_1^2q - 2h_1^2 - h_1q + 2h_1 - q)[h_1^2(3q^2 - 6q + 4) + 3h_1q(q - 6) + 3q^2]^{1/2}\sqrt{h_1}$$

Here b and a are defined in terms of $h_1 = H$ and q by the first expressions in (3.13).

The coefficients of Eq. (3.13) and its solution of the form (3.14) and (3.15) were obtained by means of the REDUCE system.

For $H < 3$ one can transfer from the optimum clearance consisting of a BESH and TES2 to a clearance consisting of two horizontal sections connected by a narrowing section without an intermediary "frozen" solution consisting of TES1 and TES2. In this case the value x_{a1} for which $h_f = 1$ at the end of the inclined section is less than 3/4. To find x_{a1} one needs to substitute $x = h = 1$ and β° from (3.8) into the equation of TES2: $h = \mathrm{H} + \beta^\circ\mathrm{H}(x - x_a)$. Solving the resulting quadratic equation for x_a and choosing the root which gives the already known value $x_a = 3/4$ for $H = 3$, we arrive at the desired formula

$$x_{a1} = \frac{2\mathrm{H}}{2\mathrm{H} - 1 + \sqrt{4\mathrm{H} - 3}}, \quad 1 \leqslant \mathrm{H} \leqslant 3 \tag{3.16}$$

The substitution of $x_a = x_{a1}$ from (3.16) into the first formula (3.6) gives the 'boundary' value $q^\circ = q_1^\circ$, and consequently $q_1 = q_1^\circ$ as a function of H,

$$q_1^\circ = \frac{2H+3+\sqrt{4H-3}}{H^2+2H+3}, \quad 1 \leqslant H \leqslant 3 \tag{3.17}$$

As follows from (3.17), $q_1^\circ(3) = 2/3$ and $q_1^\circ(1) = 1$, which corresponds to $q_1(3) = 2$ and $q_1(1) = 1$. Finally, the value β_1°, corresponding to x_{a1} from (3.16), is equal to

$$\beta_1^\circ = (2-3H-H\sqrt{4H-3})/(2H), \quad 1 \leqslant H \leqslant 3$$

In the IP a special role is played by the RP, which determines the maximum value $C_N = C_{NR}$ given as an isoperimetric condition. First, the solution of the RP can be obtained from the corresponding equations and conditions in Section 2, including (2.9), i.e. the condition that the multiplier λ in the section at the jump of h should equal zero. For $x < x_d$ the solution of the RP involves a horizontal TES1, over which $h \equiv h_R = 3q_R/2$ by (2.10). Here, as in C_{NR}, the subscript R is assigned to the optimum values in the RP. For $x < x_d$ the RP also involves a horizontal BES1 with $h \equiv 1$. Over each of these two segments the derivatives π' and λ' are constant by (1.3) and (2.3) with $\alpha = 0$ and $\beta = 1$. Therefore π and λ are linear functions of x depending on x_d and h_R or $q_R = 2h_R/3$, which are determined by (3.6). In the RP they reduce to the equalities

$$h_R\omega^2 = 1, \quad (2h_R - 3)h_R\omega = 1 \tag{3.18}$$

with ω obtained from (3.12) by replacing x_a by x_d. The elimination of ω or h_R in (3.18) gives a cubic equation. Solving it and taking the root $h_R > 1$, we obtain

$$h_R = \frac{2+\sqrt{3}}{2} \approx 1.866, \quad x_{dR} = \frac{3+2\sqrt{3}}{9} \approx 0.7182, \quad q_R = \frac{2+\sqrt{3}}{3} \approx 1.244$$
$$q_R^\circ = \frac{2}{3}, \quad C_{NR} \approx 0.03438, \quad C_{DR} \approx 0.1409 \tag{3.19}$$

The same Eq. (3.18) and values h_R, x_{dR} and q_R can be obtained if we put $\varepsilon = 0$ in system (3.12) of the IP, which corresponds to taking the Lagrange multiplier β to be minus infinity. This means that one can transfer from the solution of the IP to a solution of the RP by continuously reducing the length and slope (its modulus) of TES2.

The RP makes sense for any constraints on h, including $H < h_R$. In such cases the optimum clearance contains, in place of TES1, also a horizontal BESH with $h = H < h_R$. The optimum x_{dRH} and q_{RH} are determined by conditions (3.6) as before, but without replacing q by $2h/2$, i.e.

$$1 = H\omega^2, \quad H - q_{RH} = (q_{RH} - 1)H^2\omega, \quad \omega = (1 - x_{dRH})H/x_{dRH}$$

Hence

$$x_{dRH} = \frac{H^{3/2}}{1+H^{3/2}}, \quad q_{RH} = \frac{H(1+\sqrt{H})}{1+H^{3/2}}, \quad 1 \leqslant H \leqslant \frac{2+\sqrt{3}}{2} \tag{3.20}$$

The formulae obtained above give the solutions of the RP and the IP for any $H \geqslant 1$. The coefficient A^h in the BES1 and BESH occurring in these formulae has the required sign.

4. The characteristics of the optimum slides constructed in accordance with the conditions obtained above are collected in Fig. 2, in which $C_N^0 = C_N/C_{NR}$ and $C_D^0 = C_D/C_{DR}$ are measured along the axes with C_{NR} and C_{DR} from (3.19). Since $C_N \leqslant C_{NR}$, it follows that $0 \leqslant C_N^0 \leqslant 1$. At the same time, C_D^0 can exceed unity because the maximum value of C_D obtained for a clearance with $h \equiv 1$, which corresponds to the IP with $H = 1$, is equal to 1/6. This gives the maximum $C_D^0 \approx 1.184$. In Fig. 2 the solid curves correspond to various values of H. Since $h_R \approx 1.866$ in the RP, all the curves corresponding to $H \geqslant h_R$ for some $C_N^0 \leqslant 1$ reach the 'envelope' (dashed curve 1), which is obtained in the IP when h has no upper bound. The five upper curves computed for $H < h_R$ do not reach the envelope and terminate when $C_N^0 < 1$. Their rightmost points correspond to the RP with the additional restriction $h \leqslant H < h_R$, while $x_{dR} = x_{dRH}$ and $q_R = q_{RH}$ are defined by (3.20). By these formulae $x_{dRH} \to 1/2$ as $H \to 1$.

The coefficients C_{N3} and C_{D3} mentioned above and the corresponding C_{N3}^0 and C_{D3}^0 turn out to be as follows:

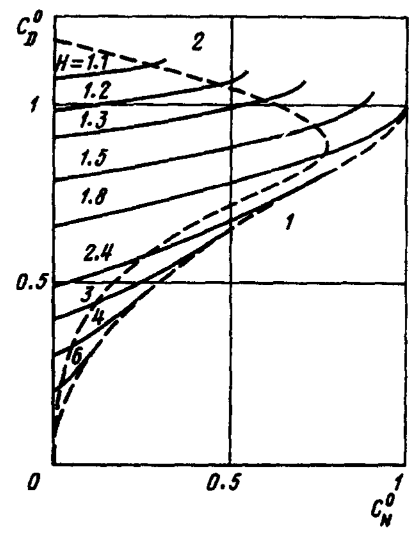

Fig. 2.

$C_{N3} \approx 0.0172$, $C_{D3} \approx 0.0916$, $C_{N3}^0 = 0.5$, $C_{D3}^0 \approx 0.650$. Therefore, by (3.10) *I*, $C_D^0 = 0.92\sqrt{(C_N^0)}$, $h_1 = 2.12\sqrt{C_N^0}$ and $0 \leqslant C_N^0 \leqslant 0.5$ for the self-modelling solution, which gives one half of envelope 1.

The optimum slides have been compared with non-optimum ones having an inclined clearance: $h(x) = 1 + (H - 1)(1 - x)$. For the "inclined" slides $h = 1$ at $x = 1$ and $h = H$ at $x = 0$, while C_N and C_D are given by

$$C_N = \frac{1}{(H-1)^2}\left(\ln H - 2\frac{H-1}{H+1}\right),\quad C_D = \frac{1}{H-1}\left(\frac{2}{3}\ln H - \frac{H-1}{H+1}\right) \tag{4.1}$$

The relation between C_D^0 and C_N^0, computed from these formulae, is given by the dashed curve 2 in Fig. 2. It does not involve the "dispersion in *H*", which does not enable us to compare C_D^0 accurately for the optimum and inclined clearances without the information given below. However, for small C_N^0, i.e. for large H, when $C_D^0 \approx 0.82\sqrt{(\ln H)}\sqrt{(C_N^0)}$ by (4.1) and (3.19), the comparison of optimum slides with C_N^0 indicates their substantial advantage.

The drags of the optimum and inclined slides have been considered for the same C_N and *H*. The results of the comparison are given below.

H	1.1	1.3	1.5	1.8	2.0	2.2	2.4	2.6	3.0	4.0	5.0	6.0	11
$C_N \times 10^3$	6	17	22	26	26	27	27	26	25	21	17	14	7
$C_D \times 10^2$	16	15	14	13	13	13	12	12	12	11	10	9	8
δC_D(%)	4	8	10	12	11	10	8	7	6	9	12	15	29

Here C_D is the drag of an inclined slid and δC_D is the amount by which it exceeds the analogous drag for the optimum slide. As a rule, the maximum height of the optimum clearance, which satisfies the condition $h \leqslant H$, is less than *H*. We can see that δC_D behaves non-monotonically. For an *H* such that the carrying capacity coefficient C_N of an inclined slide is close to the maximum, the advantage of the optimum slide as expressed by C_D is of the order of 10% and it increases rapidly as C_N^0 decreases when $C_N^0 < 0.5$.

The optimum slides have a clearance consisting either of two horizontal segments by an inclined one, or (for small C_N^0) of two segments: the entry segment horizontal and the exit segment inclined. The information on the geometry is collected in Figs 3–6. For *H* from 1.1 to 6, Fig. 3 gives the coordinate x_a of the beginning of the inclined segment, as a function of C_N^0, while Fig. 4 gives the coordinate x_b of its end. For large *H* the optimum coordinate x_a varies strongly (from 0.75 to 1) only for small C_N^0. Thus, if $H \geqslant 5$, the values $0.75 \leqslant x_a \leqslant x_{dR} \approx 0.718$ correspond to $0.2 \leqslant C_N^0 \leqslant 1$. For *H* close to and smaller than $h_R \approx 1.866$, x_a varies considerably over the whole range of values of C_N^0.

The coordinate x_b as a function of C_N^0, starting from the value corresponding to the appearance of BES1 (in Fig. 4 this corresponds to the point where the curve reaches the horizontal line $x_b = 1$), varies quite strongly. However, in this case also the curves $x_b = x_b(C_N^0, \mathrm{H})$ for $\mathrm{H} \geqslant h_R$ are either the same or close to one another.

The minimum clearance height h_b exceeds the minimum admissible value $h_b = 1$ in the neighbourhood of $C_N^0 = 0$ (Fig. 5). For $\mathrm{H} \geqslant h_R$ this neighbourhood reaches $C_N^0 = 0.5$. By what has been said above, it is independent of *H* for $\mathrm{H} \geqslant 3$. For $\mathrm{H} < h_R$ the size of this neighbourhood and the value h_b itself decrease as *h* tends to unity. In Fig. 5 *H* for each curve is equal to its ordinate when $C_N^0 = 0$.

If $\mathrm{H} \geqslant h_R$, then by the aforesaid $h = h_1$ decreases to $h_1 = h_R$ in the same way for each value *H* when a certain value C_N^0, which depends on *H*, is exceeded (Fig. 6, curve 1). For each $H > h_r$ the function $h_1 = h_1(C_N^0, \mathrm{H})$ is given

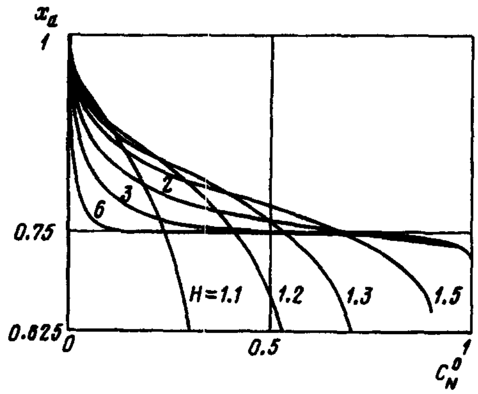

Fig. 3.

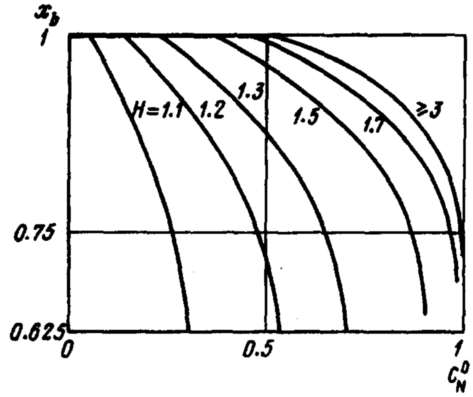

Fig. 4.

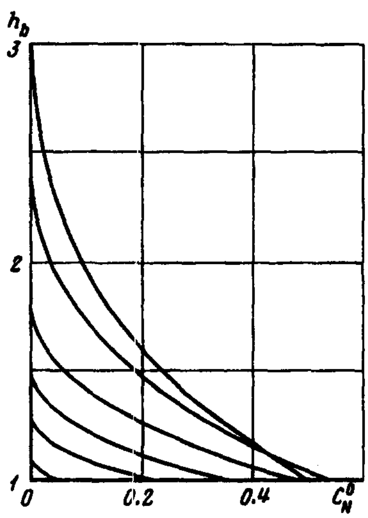

Fig. 5.

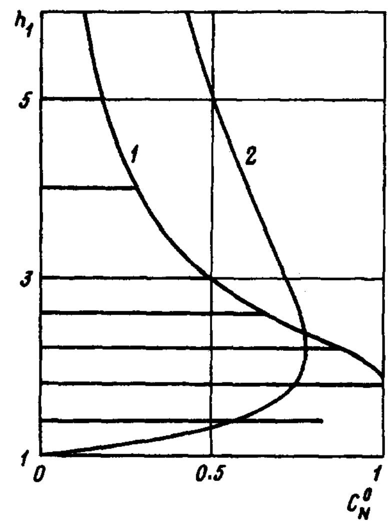

Fig. 6.

by the horizontal line segment with $h_1 \equiv$ H and, after its intersection with curve 1, by the curve itself. But if H $\leqslant h_R$, the optimum value of h_1 is equal to H for all C_N^0 attainable for such H. The two lower horizontal lines in Fig. 6 correspond to this case. Finally, curve 2 gives H as a function of C_N^0 for an inclined slide. Curves 1 and 2 in Fig. 6 along with the information on the shape of the optimum slides for large h enable us to find δC_D for any C_N^0 and H.

The above analysis and the results presented correspond to $C_N \geqslant 0$. There are possible applications in which N and C_N are negative. It can be shown that in the RP the clearance which gives the minimum of the negative coefficient C_N, i.e. the maximum of its modulus, can be obtained as the mirror image about the y axis of the clearance in the RP with $N > 0$. In the IP with given $C_{N1} > 0$ the optimum clearance can be obtained by the same reflection from the clearance in the IP with $C_N = |C_{N1}|$. In this case the dependence of C_D^0 on C_N^0 and on H is given by the curves in Fig. 2.

5. We shall begin a comparison of the results obtained in the present paper and in [10] with the fact that in [10] the full drag of the slide is called the "friction force". Next, $R = C_D/C_N$ and $R^0 = C_D/\sqrt{C_N}$ for which the clearance is optimized in [10] are special cases of the non-linear functional $F = F(C_D, C_N)$. Let F_D and F_N be the derivatives of F with respect to C_D and C_N, respectively, let $r = F_N/F_D$, and, as in [10], let $F_D > 0$ for $C_N > 0$. Then in the solution of the problem for the minimum of F, i.e. the "F problem", as a Lagrange problem the variation of the Lagrange functional, apart from an insignificant factor, is

$$\delta J = \delta C_D + r(C_D, C_N)\delta C_N + \Lambda$$

On the other hand, from the IP

$$\delta J = \delta C_D + \beta\delta C_N + \Lambda$$

We can see that, first, the minimum conditions for F can be obtained from the conditions in Sections 2 and 3 by replacing β by r. In addition, for $F_D < 0$ the inequality signs change in the conditions for a BES. Second, for $F_N \not\equiv 0$ and $F_D \not\equiv 0$ an inclined TES2 with $h' = 3r$ is possible along with TES1 in the F problem. For $F = R$ when $r = -R$, the presence of such a segment was first established in [10], which is the most important result of that paper.

Finally, the solution of the IP for which $\beta = r(C_N, C_D)$ corresponds to the solution of the F problem. For $F = R$ this occurs at a single point of curve 1 in Fig. 2 with $C_N^0 \approx 0.9673$, $C_D^0 \approx 0.9427$, $h_1 \approx 2.0024$, $x_a \approx 0.7342$, $h_b = 1$, $x_b \approx 0.8179$ and $R = 3.994$. In the problem with $F = R^0$, when $r = -R/2$, $\beta = r$ over the initial segment of curve 1, where $0 \leqslant C^0{}_N \leqslant 0.5$. On this segment $R^0 \approx 0.699$, $x_a = 0.75$, $x_b = 1$ and $h_1/h_b = 3$.

Taking into account the differences in the definitions (for example, $R^0 = C_D\sqrt{(/C_N)}$) in [10]), the above values are practically the same as those found in [10] by the "direct method". This is natural, since in [10] the structure of the optimum clearance was established using the optimality conditions and the numerical search reduces to determining x_1, x_b and h_1/h_b which give the minimum of R or R^0. It does not matter than in [10] these conditions were obtained within a "non-local" framework, which differs from the one generally adopted. It is more important to understand why the solutions of the IP found in [10] gave only those optimum clearances of the whole manifold that correspond to the lower half ($C_N \leqslant 0.5$) of curve 1 in Fig. 2. The reason is related to the way the isoperimetric condition on N is satisfied in [10] by choosing the minimum measurable clearance height h_N. Unlike h_N, the minimum admissible clearance height h_m is defined by an argument of a physical nature (the roughness of the surface, the presence of solid particles in the lubricant, possible oscillations of the slide, etc.). Therefore, by the formulation of the problem, BES1 with $h = 1$ is possible for a given value h_m, and admissible $\delta h \geqslant 0$. But if the height h is considered relative to h_N, then δh of any sign is admissible for $h = 1$ and BES1 cannot occur. As a result, from the whole manifold of solutions of the IP the minimization of R^0 gives only the self-modelling solution (3.10), i.e. the lower half of curve 1 in Fig. 2. Even though the segment $h = 1$ is introduced when the minimum of R^0 is sought in [10], the fact that its "optimum" length, equal to 0.001, is non-zero is due solely to computational errors. Finally, the absence of an upper bound for h in [10] excludes solutions with a BESH.

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